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# STRESS CONCENTRATION NEAR AN ELLIPTICAL HOLE IN A STEEP SPHERICAL SHELL

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ABSTRACT. Solution of the stress concentration problem for steep spherical shells with an arbitrary elliptical hole. Equations are derived that satisfy any boundary conditions over the elliptical hole perimeter. For the arbitrary constants of integration, a regular infinite set of algebraic equations is constructed and is solved by the method of successive approximation.

Problems of stress concentration near an elliptical hole in a spherical shell have been considered in references 8, 7, 2. It was assumed that the hole was small enough so that the state of stress in its neighborhood could be described by equations from the theory of flat shells. Also the method of solution proposed in reference 8 is applicable if the ellipse has small eccentricity. /23*

The present work considers the analogous problem for a steep spherical shell with an arbitrary elliptical hole.

The problem of the state of stress in a steep spherical shell is reduced to the solution of four monotypic differential equations of the form (see ref. 1, page 226)

$$\nabla^2\tilde{w}_j + \mu_j w_j = 0 \quad (j = 0, 1, 2) \tag{1}$$

$$\nabla^2\chi + 2\chi = 0, \tag{2}$$

where

$$\mu_0 = 2, \quad \mu_{1,2} = 1 \pm i\frac{2R}{\delta}\sqrt{3(1-\nu^2)}, \tag{3}$$

$$\nabla^2 = \frac{R^2}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)\right]. \tag{4}$$

If the solutions of equations (1) and (2) are obtained, then the flexure w is equal to:

$$w = w_0 + w_1 + w_2. \tag{5}$$

All the components of the state of stress and deformation are determined in terms of the functions $w=w(\alpha,\beta)$ and $\chi=\chi(\alpha,\beta)$.

In solving the problem of stress concentration around a hole in shells, it is convenient to select a system of coordinates such that one of the coordinate lines coincides with the contour of the hole. Therefore we associate the mean surface of the spherical shell with a special orthogonal system of

*Numbers in the margin indicate original pagination in the foreign text.

curvilinear coordinates $\alpha, \beta$ which express the Cartesian coordinates x, y, z in the following manner:

$$\begin{aligned} x &= R\frac{k}{\mathrm{dn}(\beta, k')}\frac{\mathrm{cn}(\alpha, k)}{\mathrm{dn}(\alpha, k)}, \quad 0 \leqslant \alpha \leqslant 4K, \\ y &= R\frac{k\,\mathrm{sn}(\beta, k')}{\mathrm{dn}(\beta, k')}\frac{k'\,\mathrm{sn}(\alpha, k)}{\mathrm{dn}(\alpha, k)}, \quad 0 \leqslant \beta \leqslant 2K' \\ z &= R\frac{\mathrm{cn}(\beta, k')}{\mathrm{dn}(\beta, k')}\frac{k'}{\mathrm{dn}(\alpha, k)}, \quad k'^2 + k^2 = 1, \end{aligned} \tag{6}$$

where R is the radius of the mean surface of the spherical shell, $\mathrm{sn}(\alpha, k)$, $\mathrm{cn}(\alpha, k)$, $\mathrm{dn}(\alpha, k)$ are elliptical functions (ref. 9), k is the modulus of the elliptical function, k' is an auxiliary modulus, K and K' are total elliptical integrals corresponding to the moduli k and k'. /24

The coordinate lines $\beta=\beta_0$=const are lines on the sphere whose projections on the plane xy are ellipses with semi-axes:

$$a = \frac{Rk}{\mathrm{dn}(\beta, k')}, \quad b = \frac{Rk\,\mathrm{sn}(\beta, k')}{\mathrm{dn}(\beta, k')}. \tag{7}$$

When $\beta=0$ we have $a=Rk$, $b=0$, when $\beta=K'$, it follows that $a=b=R$, i.e. the ellipse in the xy plane degenerates into a circle with radius R. The coefficients of the first quadradic form in the coordinate system (6) are equal to:

$$A = B = Rk\sqrt{\frac{1}{\mathrm{dn}^2(\beta, k')} - \frac{\mathrm{cn}^2(\alpha, k)}{\mathrm{dn}^2(\alpha, k)}}. \tag{8}$$

The equations (1) and (2) in the coordinate system (6) are written in the form

$$\frac{\partial^2\Phi_j}{\partial\alpha^2} + \frac{\partial^2\Phi_j}{\partial\beta^2} + \mu_j k^2\left[\frac{1}{\mathrm{dn}^2(\beta, k')} - \frac{\mathrm{cn}^2(\alpha, k)}{\mathrm{dn}^2(\alpha, k)}\right]\Phi_j = 0. \quad (j = 0, 1, 2). \tag{9}$$

When j=0 the function $\Phi_0$ represents the functions $w_0$ and $\chi$ which differ in that $w_0$ is an even function while $\chi$ is an odd function.

To solve equation (9) we use the method of separation of variables. The particular solution of equation (9) is represented in the form:

$$\Phi = y_1(\alpha)\cdot y_2(\beta). \tag{10}$$

Separating the variables we obtain:

$$\frac{d^2y_1}{d\alpha^2} + \left[\lambda - \mu k^2\frac{\mathrm{cn}^2(\alpha, k)}{\mathrm{dn}^2(\alpha, k)}\right]y_1 = 0, \tag{11}$$

$$\frac{d^2y_2}{d\beta^2} - \left[\lambda - \mu k^2 \frac{1}{\mathrm{dn}^2(\beta, k')}\right] y_2 = 0, \qquad (12)$$

$$\frac{1}{\mathrm{dn}^2(\beta, k')},$$

where $\lambda$ is the separation constant.

Equation (11) is one form of the Lamé differential equation (ref. 9) because

$$\frac{\mathrm{cn}(\alpha, k)}{\mathrm{dn}(\alpha, k)}\Big|_{\alpha=i\beta} = \frac{1}{\mathrm{dn}(\beta, k')},$$

equation (12) is obtained from equation (11) by changing variables

$$\alpha = i\beta. \qquad (13)$$

Thus if solution $y_1(\alpha)$ is known then solution $y_2(\beta)$ is obtained by changing variables (13).

Since $\alpha$ varies in the limits $0 \leq \alpha \leq 4K$ and $\Phi$ must be a periodic function of $\alpha$ the solution $y_1(\alpha)$ must also be a periodic function of $\alpha$ with a period 2K or 4K.

Thus we have transformed to a problem of eigenvalues and eigenfunctions for the Lamé differential equation with periodic boundary conditions. /25

References 5, 9 consider the problem of the Lamé equation in the case when

$$\mu = n(n+1). \qquad (14)$$

Here n is a whole number and the established eigenfunctions are called the Lamé functions.

Below we present a method for finding the eigenvalues and the eigenfunctions of the differential operator:

$$\left.\begin{aligned} &\frac{d^2y_1}{d\alpha^2} + \left[\lambda - \mu k^2 \frac{\mathrm{cn}^2(\alpha, k)}{\mathrm{dn}^2(\alpha, k)}\right] y_1 = 0, \\ &y_1(0) = y_1(2K), \\ &y_1'(0) = y_1'(2K), \\ &\qquad \text{or} \\ &y_1(0) = y_1(4K), \\ &y_1'(0) = y_1'(4K). \end{aligned}\right\} \qquad (15)$$

It follows from the periodic properties of the Jacobi elliptic functions (ref. 9) that

$$\frac{\mathrm{cn}(\alpha, k)}{\mathrm{dn}(\alpha, k)} = \mathrm{sn}(\alpha^*, k)\Big|_{\alpha^* = \alpha + K}. \qquad (16)$$

There also exists an expansion (ref. 9) of the following function in Fourier series

$$k^2 sn^2(\alpha^*, k) = \frac{K-E}{K} - \frac{2\pi^2}{K^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos\frac{n\pi\alpha^*}{K}. \tag{17}$$

Here E is the total elliptical integral of the second kind while the parameter q is equal to

$$q = e^{-\pi\frac{K}{K'}}. \tag{18}$$

It follows from (16) and (17) that

$$k^2\frac{cn^2(\alpha, k)}{dn^2(\alpha, k)} = \frac{K-E}{K} - \frac{2\pi^2}{K^2}\sum_{n=1}^{\infty}(-1)^n\frac{nq^n}{1-q^{2n}}\cos\frac{n\pi\alpha}{K}. \tag{19}$$

This series converges over the entire band

$$\left|Im\frac{\pi\alpha}{2K}\right| < \frac{1}{2}\pi Im\tau, \quad \tau = i\frac{K'}{K}. \tag{20}$$

Taking into account relationship (19) we can write the differential equation in (15) in the form:

$$y_1'' + \left[\lambda - \frac{K-E}{K}\mu + \frac{2\pi^2\mu}{K^2}\sum_{n=1}^{\infty}(-1)^n\frac{nq^n}{1-q^{2n}}\cos\frac{n\pi d}{K}\right]y_1 = 0. \tag{21}$$

In the subsequent investigations it is convenient to make a change in the variables:

$$\frac{\pi\alpha}{2K} = \varphi, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Then we obtain the following differential operator: /26

$$\left.\begin{aligned} &y_1'' + \left[a - 8\mu\sum_{n=1}^{\infty}(-1)^{n-1}\frac{nq^n}{1-q^{2n}}\cos 2n\varphi\right]y_1 = 0, \\ &\qquad y_1(0) = y_1(\pi), \\ &\qquad y_1'(0) = y_1'(\pi), \\ &\text{or} \\ &\qquad y_1(0) = y_1(2\pi), \\ &\qquad y_1'(0) = y_1'(2\pi), \end{aligned}\right\} \tag{22}$$

where

$$a = \left(\lambda - \frac{K-E}{K}\mu\right)\frac{4K^2}{\pi^2}.$$

When the change (13) is made equation (12) will have the form:

$$\frac{d^2y_2}{d\psi^2} - \left[a - 8\mu \sum_{n=1}^{\infty} (-1)^{n-1} \frac{nq^n}{1-q^{2n}} \operatorname{ch} 2n\psi\right] y_2 = 0, \tag{23}$$

where

$$\psi = \frac{\pi\beta}{2K}. \tag{24}$$

From inequality (20) we find that

$$0 \leqslant \psi < \frac{\pi K'}{2K}. \tag{25}$$

Thus if we assume that $\psi$=0 is a pole, then the solutions $y_2(\psi)$ obtained by the method described above are determined in the entire upper region of the hemisphere with the exception of a boundary circle with radius R.

The differential operator (22) is the Hill differential operator (ref. 4). The eigenfunctions and eigenvalues of this operator can be obtained by using a method from the theory of perturbations (ref. 3).

Let us transform the differential equation for the operator (22) by isolating its principal part:

$$y_1'' + [a - 2h\cos 2\varphi + \sum_{n=1}^{\infty} (-1)^n g_n q^n] y_1 = 0, \tag{26}$$

where

$$h = \frac{4\mu q}{1-q^2}, \quad g_1 = 0, \quad g_n = \frac{8\mu n}{1-q^{2n}} \cdot \cos 2n\varphi. \tag{27}$$

For the unperturbed differential operator we take the Mathieu differential equation with the same periodic boundary conditions, specifically:

$$\Psi_n^{(0)''} + (\varkappa_n^{(0)} - 2h\cos 2\varphi)\,\Psi_n^{(0)} = 0, \tag{28}$$

$$\left.\begin{aligned} \Psi_n^{(0)}(0) &= \Psi_n^{(0)}(\pi), \\ \Psi_n^{(0)\prime}(0) &= \Psi_n^{(0)\prime}(\pi), \\ &\text{or} \\ \Psi_n^{(0)}(0) &= \Psi_n^{(0)}(2\pi), \\ \Psi_n^{(0)\prime}(0) &= \Psi_n^{(0)\prime}(2\pi). \end{aligned}\right\} \tag{29}$$

The function $\Psi_n^{(0)}$ is any one of four possible Mathieu functions:

$$\left.\begin{aligned} ce_{2n}(\varphi) &= \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2r\varphi, \\ ce_{2n+1}(\varphi) &= \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos (2r+1)\varphi, \\ se_{2n+1}(\varphi) &= \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin (2r+1)\varphi, \\ se_{2n+2}(\varphi) &= \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin (2r+2)\varphi. \end{aligned}\right\} \qquad (30)$$

By using the method of the theory of perturbations (ref. 3), we represent the eigenvalues and the eigenfunctions of the operator (22) in the form /27

$$\left.\begin{aligned} a_n &= \varkappa_n^{(0)} + \sum_{m=1}^{\infty} q^m \varkappa_n^{(m)}, \\ y_1^{(n)} &= \Psi_n^{(0)}(\varphi) + \sum_{m=1}^{\infty} q^m \Psi_n^{(m)}(\varphi), \end{aligned}\right\} \qquad (31)$$

where $\varkappa_n^{(0)}, \Psi_n^{(0)}$ are eigenfunctions and eigenvalues of the unperturbed operator (28)-(29).

If we substitute representation (31) into equation (26) and set the coefficients in front of the identical powers q equal to zero, we obtain an infinite sequence of nonhomogeneous differential equations for the unknown functions $\Psi_n^{(m)}(\varphi) = \Psi_n^{(m)}$:

$$\Psi_n^{(m)''} + [\varkappa_n^{(0)} - 2h\cos 2\varphi]\,\Psi_n^{(m)} = -\sum_{p=1}^{m} [(-1)^p g_p + \varkappa_n^{(p)}]\,\Psi_n^{(m-p)}. \qquad (32)$$

Two cases are possible:

a) h is a real number.

Since the homogeneous problem (28)-(29) has a nonzero solution, the nonhomogeneous problem can be solved only in the case when (ref. 3)

$$\int_0^{2\pi} \sum_{p=1}^{m} [(-1)^p g_p + \varkappa_n^{(p)}]\,\Psi_n^{(m-p)}\Psi_n^{(0)} d\varphi = 0. \qquad (33)$$

Condition (33) may be satisfied by a corresponding selection of the values $\varkappa_n^{(p)}$.

b) h is a complex number.

In this case problem (28)-(29) is not self-conjugate (ref. 6) and its corresponding nonhomogeneous problem (32)-(29) is solvable only in the case when

$$\int_0^{2\pi} \sum_{p=1}^{m} [(-1)^p g_p + \varkappa_n^{(p)}] \Psi_n^{(m-p)} \overline{Z_n^{(0)}}\, d\varphi = 0, \tag{34}$$

where $Z_n^{(0)}$ are eigenfunctions of the operator which is conjugate to (28)-(29),

$$Z_n^{(0)\prime\prime} + (\varkappa_n^{(0)} - 2h\cos 2\varphi)\, Z_n^{(0)} = 0, \tag{35}$$

$$\left.\begin{aligned} Z_n^{(0)}(0) &= Z_n^{(0)}(\pi), \\ Z_n^{(0)\prime}(0) &= Z_n^{(0)\prime}(\pi), \\ &\text{or} \\ Z_n^{(0)}(0) &= Z_n^{(0)}(2\pi), \\ Z_n^{(0)\prime}(0) &= Z_n^{(0)\prime}(2\pi). \end{aligned}\right\} \tag{36}$$

Problem (32)-(29) is simplified if h is a purely imaginary number. This can be assumed with the known degree of accuracy in the theory of thin shells if in the second equation (3) we neglect unity compared with $\frac{2R}{\delta}\sqrt{3(1-\nu^2)}$. /28

It follows from (28) and (35) that:

$$Z_n^{(0)} = \overline{\Psi_n^{(0)}}; \quad \Psi_n^{(0)} = \overline{Z_n^{(0)}}. \tag{37}$$

From equation (32) we obtain the following for the function $\Psi_n^{(1)}$:

$$\Psi_n^{(1)\prime\prime} + [\varkappa_n^{(0)} - 2h\cos 2\varphi]\, \Psi_n^{(1)} = -(g_1 - \varkappa_n^{(1)})\, \Psi_n^{(0)}.$$

Because $g_1=0$ it follows from (33) that:

$$\varkappa_n^{(1)} = 0, \quad \Psi_n^{(1)} = 0. \tag{38}$$

We note that an analogous result is obtained from (34). For the function $\Psi_n^{(2)}$ we obtain the equation:

$$\Psi_n^{(2)\prime\prime} + [\varkappa_n^{(0)} - 2h\cos 2\varphi]\, \Psi_n^{(2)} = -[(-g_1 + \varkappa_n)\, \Psi_n^{(1)} + (g_2 + \varkappa_n^{(2)})\, \Psi_n^{(0)}]. \tag{39}$$

We expand the product $g_2\Psi_n^{(0)}$ in series of functions $\Psi_n^{(0)}$:

$$g_2\Psi_n^{(0)} = \sum_{s=0}^{\infty} d_s^{(2,n)}\Psi_s^{(0)}. \tag{40}$$

Here

$$d_s^{(2)} = \frac{1}{N_n}\int_0^{2\pi} g_2 \Psi_n^{(0)} \overline{Z_s^{(0)}}\, d\varphi = \frac{1}{N_n}\int_0^{2\pi} g_2 \Psi_n^{(0)} \Psi_s^{(0)}\, d\varphi,$$
$$N_n = \int_0^{2\pi} \Psi_n^{(0)} \overline{Z_n^{(0)}}\, d\varphi = \int_0^{2\pi} [\Psi_n^{(0)}]^2\, d\varphi. \qquad (41)$$

Substituting (40) into (39), we find

$$\varkappa_n^{(2)} = -d_n^{(2,n)},$$
$$\Psi_n^{(2)} = -\sum_{s\neq n} \frac{d_s^{(2,n)}}{\varkappa_n^{(0)} - \varkappa_s^{(0)}} \Psi_s^{(0)}. \qquad (42)$$

We obtain the following expression for the function $\Psi_n^{(3)}$:

$$\Psi_n^{(3)''} + (\varkappa_n^{(0)} - 2h\cos 2\varphi)\Psi_n^{(3)} = -[(-g_1 + \varkappa_n^{(1)})\Psi_n^{(2)} + (g_2 + \varkappa_n^{(2)})\Psi_n^{(1)} + (-g_3 + \varkappa_n^{(3)})\Psi_n^{(0)}].$$

As in the preceding case we represent

$$g_3 \Psi_n^{(3)} = \sum_{s=0}^{\infty} d_s^{(3,n)} \Psi_n^{(0)} \qquad (43)$$

and find

/29

$$\varkappa_n^{(3)} = d_n^{(3,n)},$$
$$\Psi_n^{(3)} = +\sum_{s\neq n} \frac{d_n^{(3,n)}}{\varkappa_n^{(0)} - \varkappa_s^{(0)}} \Psi_n^{(0)} \qquad (44)$$

Thus in accordance with (31) and (30) the eigenfunctions of the differential operator (22) may be represented in the series of Mathieu functions.

The solution $y_2^{(n)}(\psi)$ of equation (23) is obtained by a simple substitution of variables in accordance with equality (13) as follows:

$$y_2^{(0)}(\psi) = \Psi_n^{(0)}(i\psi) + \sum_{m=1}^{\infty} q^m \Psi_n^{(m)}(i\psi). \qquad (45)$$

The region where solution (45) is valid is determined by inequality (25).

By using the constructed solutions (31) and (45) we can satisfy any boundary conditions on the contour of an elliptical hole in the shell. In this

case for arbitrary constants of integration we ob ain a regular infinite system of algebraic equations which is solved by the method of successive approximations.

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